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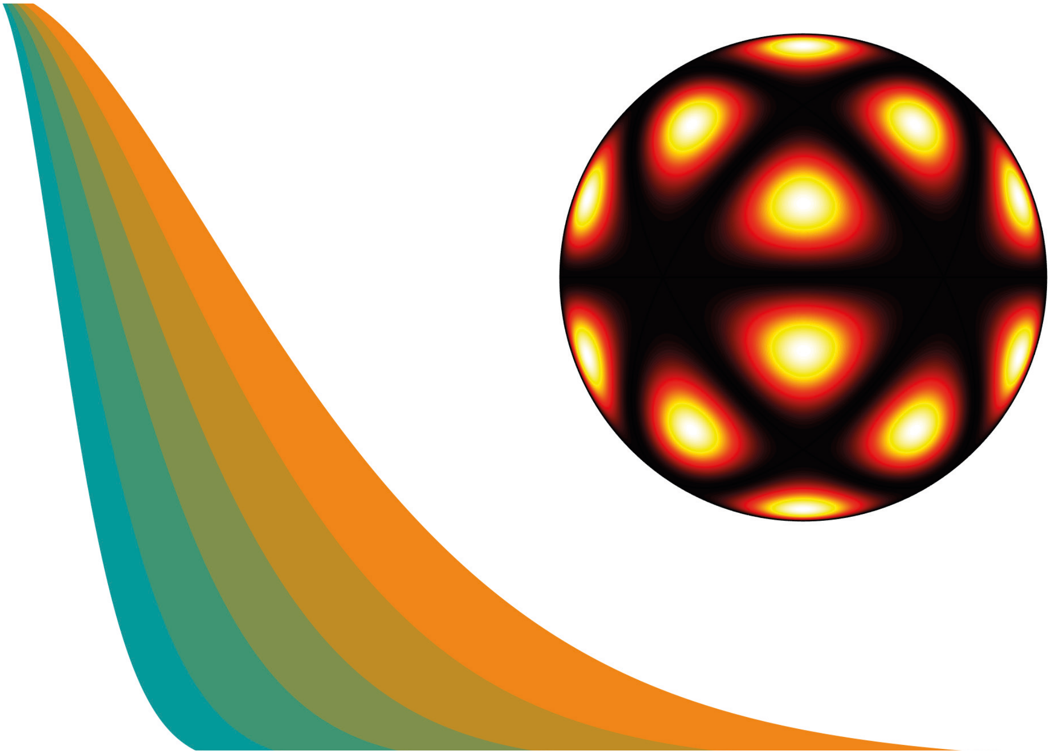
Extreme points of the Vandermonde determinant in numerical approximation, random matrix theory and financial mathematics

Asaph Keikara Muhumuza



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Akademisk avhandling

som för avläggande av filosofie doktorsexamen i matematik/tillämpad matematik vid Akademin för utbildning, kultur och kommunikation kommer att offentligen försvaras måndagen den 14 december 2020, 15.15 i Lambda +(digitalt Zoom), Mälardalens Högskola, Västerås.

Fakultetsopponent: Docent Olga Liivapuu, Estonian University of Life Sciences

MÄLARDALEN UNIVERSITY
SWEDEN

Akademin för utbildning, kultur och kommunikation

Abstract

This thesis discusses the extreme points of the Vandermonde determinant on various surfaces, their applications in numerical approximation, random matrix theory and financial mathematics. Some mathematical models that employ these extreme points such as curve fitting, data smoothing, experimental design, electrostatics, risk control in finance and method for finding the extreme points on certain surfaces are demonstrated.

The first chapter introduces the theoretical background necessary for later chapters. We review the historical background of the Vandermonde matrix and its determinant, some of its properties that make it more applicable to symmetric polynomials, classical orthogonal polynomials and random matrices.

The second chapter discusses the construction of the generalized Vandermonde interpolation polynomial based on divided differences. We explore further, the concept of weighted Fekete points and their connection to zeros of the classical orthogonal polynomials as stable interpolation points.

The third chapter discusses some extended results on optimizing the Vandermonde determinant on a few different surfaces defined by univariate polynomials. The coordinates of the extreme points are shown to be given as roots of univariate polynomials.

The fourth chapter describes the symmetric group properties of the extreme points of Vandermonde and Schur polynomials as well as application of these extreme points in curve fitting.

The fifth chapter discusses the extreme points of Vandermonde determinant to number of mathematical models in random matrix theory where the joint eigenvalue probability density distribution of a Wishart matrix when optimized over surfaces implicitly defined by univariate polynomials.

The sixth chapter examines some properties of the extreme points of the joint eigenvalue probability density distribution of the Wishart matrix and application of such in computation of the condition numbers of the Vandermonde and Wishart matrices.

The seventh chapter establishes a connection between the extreme points of Vandermonde determinants and minimizing risk measures in financial mathematics. We illustrate this with an application to optimal portfolio selection.

The eighth chapter discusses the extension of the Wishart probability distributions in higher dimension based on the symmetric cones in Jordan algebras. The symmetric cones form a basis for the construction of the degenerate and non-degenerate Wishart distributions.

The ninth chapter demonstrates the connection between the extreme points of the Vandermonde determinant and Wishart joint eigenvalue probability distributions in higher dimension based on the boundary points of the symmetric cones in Jordan algebras that occur in both the discrete and continuous part of the Gindikin set.

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Västerås, November, 2020
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This thesis is dedicated to my beloved parents
Mr. Emmanuel Keikara and Mrs. Jane Keikara

Popular Science Summary

Mathematics, naturals sciences and technology are strongly interrelated both in theory and practice. Mathematical theories like analysis, geometry and algebra are all crucial components of mathematical models in many applications. Mathematical models are mainly applied in natural sciences that include physics, biology, earth-science and chemistry, and in technological disciplines including computer science and telecommunication engineering, electrical, mechanical and chemical engineering, as well as in the social-economic science disciplines that include economics, finance, operations research, psychology, sociology and political sciences. The most important thing to note is that a wide variety mathematical models whether linear or non-linear, static or dynamic, explicit or implicit, discrete or continuous, deterministic or stochastic (or probabilistic), strategic or non-strategic and deductive or inductive as used in various science disciplines can all be constructed based on the concept of matrix theory.

In this thesis a special matrix called the Vandermonde matrix is our main focus in studying certain mathematical models in numerical analysis, random matrix theory and random field based on optimization the Vandermonde determinant. Here, mathematical optimization a mathematical programming principle mainly refers to the systematic criteria of selection of a best optimal (or extreme) elements, from some set of available large field of alternative points represented in a matrix form and such elements should maximize or minimize the determinant of the same matrix.

Most mathematical models are characterized by the phenomenon of well–posedness whereby, for example, according to Jacques Hadamard a mathematical model of physical phenomenon is said to be well-posed problem if it has the properties that the solution exists, the solution is unique and the solution's behaviour changes continuously with the initial conditions. In continuum models that must often require to be discretized in order to obtain a numerical solution, whereas the solutions may be continuous with respect to the initial conditions, they may suffer from numerical instability when solved with finite precision, or with errors in the data. Much as the problem may be well–posed, it may still suffer to be ill–conditioned,due to the fact that a small error in the initial data can result in even much larger errors in the final solution. This fact of stability of solutions inspired our study of the Vandermonde matrix and optimization of its determinant a technique that is highly employed in error control for ill-conditioned problem and also indicated by a large condition number.

The study of extreme points of Vandermonde determinant and conditioning inspired us to extend the results to investigate such systems including Coulomb's system and energy level spacing for heavy nuclear atoms which are characterised by joint eigenvalue distribution also called ensembles that occur mainly in random matrix theory and random fields. These extreme points of the Vandermonde determinant are seen to play a significant role in both physical and biological science based on the zeros of the classical orthogonal polynomials, the Gaussian ensembles and the Wishart ensembles in symmetric cones of Jordan algebras.

Populärvetenskaplig Sammanfattning

Matematik, naturvetenskap och teknologi är starkt sammankopplade både i teori och praktik. Matematiska områden såsom analys, geometri och algebra är kritiska komponenter i konstruktionen av matematiska modeller inom många tillämpningsområden. Matematiska modeller används främst i naturvetenskaper såsom fysik, biologi, geovetenskap och kemi, och inom teknologiska områden såsom datorvetenskap, telekommunikation, elektroteknik, mekanik och kemiteknik, men även inom social-ekonomiska områden såsom ekonomi, finans, operationsanalys, psykologi, sociologi och statsvetenskap. Det som är viktigast att ha i åtanke är att de flesta matematiska modeller, oavsett om de är linjära eller icke-linjära, statiska eller dynamiska, explicita eller implicita, diskreta eller kontinuerliga, deterministiska eller stokastiska (slumpmässiga), strategiska eller ostrategiska, baserade på deduktion eller induktion, kan alla konstrueras baserat på begrepp från matristeori.

I denna avhandling är en speciell slumpmässig matris som kallas för Vandermondematrisen vårt huvudfokus, vi kommer att studera vissa matematiska modeller från numerisk analys, teorin om slumpmässiga matriser och slumpmässiga kroppar baserat på optimering av Vandermonde determinanten. Med matematisk optimering menar vi här systematiskt urval av de mest optimala (eller mest extrema) element från någon stor kropp av möjliga punkter som representeras i matrisform på så sätt att dessa element maximerar eller minimerar determinanten av samma matris.

De flesta modeller ger problem som kan sägas vara väl-ställda, med detta menas, enligt t.ex. Jaques Hadamard, att en matematiska modell av ett fysikaliskt fenomen get välställda problem om problemets lösning existerar, lösningen är entydig och lösningens beteende ändras kontinuerligt om problemets initialvillkor ändras. I kontinuerliga modeller som behöver diskretiseras för att kunna behandlas med numeriska metoder, så kan det vara så att medan lösningen ändras kontinuerligt med avseende på initialvillkoren, så introducerar begränsningar i numerisk precision instabilitet i lösningen. På liknande sätt kan fel i data introducera instabilitet. Stabiliteten av lösningar inspirerade vår undersökning av Vandermondematrisen och metoder för optimering av dess determinant då detta är relevant för felkontroll för dålig ställda problem på grund av kopplingar mellan determinanten och matrisen konditionstal.

Studien av extrempunkter hos Vandermondedeterminanten och kondition insperade vidare undersökning av systems såsom Coulombs system och avstånd mellan energinivåer för tunga kärnpartiklar vilka beskrivs av egenvärdena för en typ av multivariat distribution som kallas för en ensemble och som ofta dyker upp i teorin för slumpmässiga matriser och slumpmässiga kroppar. Extrempunkterna för Vandermondedeterminanten kan beskrivas med hjälp av nollställena till klassiska ortogonala polynom för den Gauss-ensemblen, Wishart-esemblen samt ensembler i den symmetriska konen av Jordan-algebror.

List of Papers

The chapters 2 through to 9 in this thesis are based, respectively, on the following list of papers:

Paper A. Muhumuza Asaph K., Lundengård Karl, Österberg Jonas, Silvestrov Sergei, Mango John M., Kakuba Godwin. *The Generalized Vandermonde Interpolation Polynomial Based on Divided Differences*, SMTDA2018 Conference Proceedings, **ISAST2018**, 443–456, 2018.

Paper B. Muhumuza Asaph K., Lundengård Karl, Österberg Jonas, Silvestrov Sergei, Mango John M., Kakuba Godwin. *Extreme points of the Vandermonde determinant on surfaces implicitly determined by a univariate polynomial*. In: Silvestrov S., Malyalenko A., Rančić M., (Eds.), Algebraic Structures and Applications. SPAS 2017. Springer Proceedings in Mathematics & Statistics, vol 317, 791–818, 2020.
https://doi.org/10.1007/978-3-030-41850-2-33.

Paper C. Muhumuza Asaph K., Silvestrov Sergei, (2019). *Symmetric Group Properties of Extreme Points of Vandermonde Determinant and Schur polynomials*. Accepted for publication in: Sergei Silvestrov, Anatoliy Malyalenko, Milica Rančić M., (Eds.), SPAS 2019: Algebraic Structures and Applications.

Paper D. Muhumuza Asaph K., Lundengård Karl, Silvestrov Sergei, Mango John M., Kakuba Godwin. *Optimization of the Wishart Joint Eigenvalue Probability Density Distribution Based on the Vandermonde Determinant*. In: Silvestrov S., Malyalenko A., Rančić M., (Eds.), Algebraic Structures and Applications. SPAS 2017. Springer Proceedings in Mathematics & Statistics, vol 317, 819–838, 2020.
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Paper E. Muhumuza Asaph K., Lundengård Karl, Silvestrov Sergei, Mango John M., Kakuba Godwin. *Properties of the Extreme Points of the Joint Eigenvalue Probability Density Function of the Wishart Matrix*. In ASMDA2019, 18th Applied Stochastic Models and Data Analysis International Conference, ISAST: International Society for the Advancement of Science and Technology. (pp. 559–571), 2019.

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Paper G. Muhumuza Asaph K., Lundengård Karl, Malyarenko Anatoliy, Silvestrov Sergei, Mango John M., Kakuba Godwin, (2019). *The Wishart Distribution on Symmetric Cones.* Accepted for publication in: Silvestrov S., Malyalenko A., Rančić M., (Eds.), SPAS2019. Algebraic Structures and Applications. SPAS 2019. Springer Proceedings in Mathematics & Statistics, 2020.

Paper H. Muhumuza Asaph K., Lundengård Karl, Malyarenko Anatoliy, Silvestrov Sergei, Mango John M., Kakuba Godwin, (2019). *Extreme Points of the Vandermonde Determinant and Wishart Ensembles on Symmetric Cones.* Accepted in: Silvestrov S., Malyalenko A., Rančić M., (Eds.), Springer International Publishing, 2020.

The other co-authored paper(s) include:

Paper I. Muhumuza Asaph K., Malyarenko Anatoliy, Silvestrov Sergei, (2017). Lie symmetries of the Black–Scholes type equations in financial mathematics, ASMDA2017 Conference Proceedings, **ISAST2017**, 723–740, 2017.

Parts of this thesis have been presented at the following international conferences and workshops:

- ASMDA2017: The 17th conference of the Applied Stochastic Models and data Analysis International Society and demographic 2017 Workshop, 11-14 June 2019, London, United Kingdom, 6th – 9th June 2017.
- SMTDA2018: The 5th Stochastic Modeling Techniques and Data Analysis International Conference, Chania, Crete, Greece, 12th – 15th June 2018.
- IWAP2018: The 9th International Workshop On Applied Probability, Budapest, Hungary, 18th - 21st June 2018.
- SPAS2017: International Conference on Stochastic Processes and Algebraic Structures-From Theory Towards Applications, Västerås/Stockholm, Sweden, 4th – 6th October, 2017.
- ASMDA2019: The 18th conference of the Applied Stochastic Models and data Analysis International Society and demographic 2019 Workshop, Florence, Italy, 11th – 14th June 2019.
- SPAS2019: International Conference on Stochastic Processes and Algebraic Structures-From Theory Towards Applications, Västerås, Sweden, 30th September – 2nd October 2019.

Notations

The following notations will be used throughout the Thesis unless defined otherwise.

$\mathbb{N},\mathbb{Z},\mathbb{R},\mathbb{C}$	–	The set of Natural, Integers, Real and Complex numbers.
$\mathbf{x},\mathbf{v}$	–	Bold, roman lower letters denote vectors.
$\mathbf{X},\mathbf{V},\mathbf{M}$	–	Bold, uppercase letters denote matrices.
δ	–	The index $\delta=(n-1,\ldots,2,1,0)$, unless stated otherwise.
λ	–	The partition $\lambda=(\lambda_1,\ldots,\lambda_n)$, unless stated otherwise.
$\mathbf{C}_{i,j},\mathbf{M}_{i,j}$	–	Element on the i–th row and j–th column of $\mathbf{M}$.
$\mathbf{M}_{\cdot,j}$	–	Column vector of all elements from the j–th column of $\mathbf{M}$.
$\mathbf{M}_{i,\cdot}$	–	Row vector of all elements from the i–th row of $\mathbf{M}$.
$[a_{ij}]^{nm}$	–	$ij-n\times m$ matrix with element a_{ij} in the i–th row and j–th column.
$\mathbf{V}_{nm}(\mathbf{x})$	–	$n\times m$ – Vandermonde matrix with respect to $\mathbf{x}\in\mathbb{R}^n$.
$\mathbf{V}(\mathbf{x})=\mathbf{V}_{nn}(\mathbf{x})$	–	n– square Vandermonde matrix with respect to $\mathbf{x}\in\mathbb{R}^n$.
$\det\big(\mathbf{V}(\mathbf{x})\big)=v_n(\mathbf{x})$	–	Determinant of the n– square Vandermonde matrix.
$\mathbf{V}_{\delta}(\mathbf{x})=\mathbf{V}_n(\mathbf{x})$	–	Vandermonde matrix with respect to index δ and $\mathbf{x}\in\mathbb{R}^n$.
$\mathbf{V}_{\lambda+\delta}(\mathbf{x})$	–	Vandermonde matrix with respect to partition λ and $\mathbf{x}\in\mathbb{R}^n$.
$\det\big(\mathbf{V}_{\delta}(\mathbf{x})\big)=a_{\lambda}(\mathbf{x})$	–	Determinant of the Vandermonde matrix with respect to index δ.
$\det\big(\mathbf{V}_{\lambda+\delta}(\mathbf{x})\big)=a_{\delta+\lambda}(\mathbf{x})$	–	Determinant of the Vandermonde matrix with respect to partition λ.
$s_{\lambda}(\mathbf{x})=a_{\delta+\lambda}(\mathbf{x})/a_{\delta+\lambda}(\mathbf{x})$	–	The Schur polynomial with respect to partition λ.
$C^k[\mathbb{K}]$	–	The continuous functions with k–th derivative on the field $\mathbb{K}$.
$\lVert\mathbf{x}\rVert_p=\left(\sum_{k=1}^{n}\lvert x_k\rvert^p\right)^{\frac{1}{p}}$	–	The p–norm of $\mathbf{x}\in\mathbb{R}^n$, where $p=2$ is the Euclidean norm.
S_n^p	–	The n–dimension p–sphere, $S_n^p(r)=\left\{\mathbf{x}\in\mathbb{R}^{n+1}\ :\ \sum_{k=1}^{n}\lvert x_k\rvert^{p+1}=r^{p+1}\right\}$.
$\lVert\cdot\rVert_{\mathrm{F}}$	–	The Frobenius–norm where $\lVert\mathbf{X}\rVert_{\mathrm{F}}=\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\lvert x_{ij}\rvert^2\right)^{\frac{1}{2}}=\sqrt{\mathrm{tr}(\mathbf{A}^{\top}\mathbf{A})}$.
$\kappa(\mathbf{X})=\lVert\mathbf{X}^{-1}\rVert\lVert\mathbf{X}\rVert$	–	The condition number of $\mathbf{X}$, where $\mathbf{X}^{-1}$ is inverse of $\mathbf{X}$.
$H_n(\cdot),P_n^{(\alpha,\beta)}(\cdot),L_n(x),P_n(\cdot)$	–	The Hermite, Jacobi, Legendre and Laguerre orthogonal polynomials.
$\Gamma(x),\beta(\cdot)$	–	The Gamma and Beta functions, $\Gamma(\alpha)=(\alpha-1)!,\ \beta(a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$.
${}_2F_2(a,b;c;x)$	–	The hypergeometric function.
$\frac{d^kf}{dx^k}=f^{(k)}(x)$	–	The k–th derivative of the function f with respect to x.
$\frac{\partial^nf}{\partial x^n}=f^{(n)}(x)$	–	The n–th partial derivative of the function f with respect to x.
$\mathbb{P}(A)$	–	The probability of event A.
$\mathbb{E}(X),\mathrm{Var}(x)=V(X)$	–	The expectation and variance of random variable X respectively.

Contents

CONTENTS

Introduction

This chapter is based on Paper **A**, Paper **B**, Paper **C**, Paper **D**, Paper **F** and Paper **G**, and gives the general overview of the contents of Chapters 2, 3, 4, 5, 6 7, 8, and 9.

Paper A. Muhumuza Asaph K., Lundengård Karl, Österberg Jonas, Silvestrov Sergei, Mango John M., Kakuba Godwin. *The Generalized Vandermonde Interpolation Polynomial Based on Divided Differences*, SMTDA2018 Conference Proceedings, **ISAST2018**, 443–456, 2018.

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